On the complexity of compact coalitional games[1]

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[1] IJCAI 2009 - Joint work with G. Greco, E. Malizia and F. Scarcello

Talk Outline

What is game theory about?

- Game theory helps in understanding how decisions are taken by rational agents (players)
- Various game models to support diverse analysis scenarios

Coalitional games

- A *coalitional game* $\mathcal{G}$:
  - N, the set of players that can form *coalitions*
  - $v : 2^N \mapsto \mathbb{R}$, worth function, assigns to each coalition $S \subseteq N$ the worth $v(S)$ which players in S obtain by cooperating
- Outcome of $\mathcal{G}$: a vector of payoffs $(x_i)_{i \in N} \in \mathbb{R}^{|N|}$, that specifies the distribution of the worth granted to each player in N

Coalitional games

- Fundamental problem: characterizing *solution concepts*, capturing most desirable outcomes (fair worth distributions)
- Issue widely addressed in the theory: tell a given solution to suitably render the intuition of fairness and stability
- Well-known and accepted solution concepts are the *stable sets*, *Shapely value*, the *core*, the *kernel*, the *bargaining set*, and the *nucleolus*

Example

- Multiple users route network traffic through a switch, which has a flow-dependent delay (cost)
- The queueing delay cost has to be shared among the users
- Users are self-motivated

Modeled as a coalitional game, a suitable solution concept (the Shapley value, in this case) establishes fair cost sharing.

Premises

- Dealing with representable games: avoid the exponential blow-up of explicitly representing 2^n worth values
- Complying with the *bounded rationality principle* that decision making cannot imply unbounded resources to support reasoning
  - Captured by assessing the amount of needed reasoning resources via complexity classes

Graph games

Deng and Papadimitriou[2] considered the setting of *graph games*. Let N be the players. A game is a weighted undirected graph $G = \langle (N, E), w \rangle$, where:

- the list w encodes the edge weighting function: $w(e) \in \mathbb{R}$ weighs the edge $e \in E$
- For a coalition $S \subseteq N$, $v(S) = \sum_{e \in E | e \subseteq S} w(e)$

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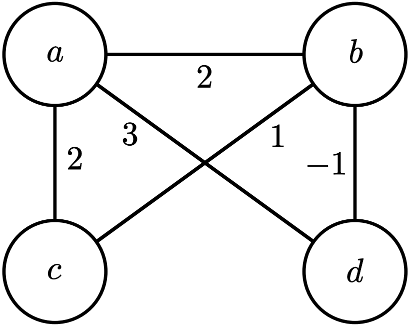
[2] On the complexity of cooperative solution concepts, *Mathematics of Operations Research*, 19(2), 1994

An example of a graph game



Worths for some sample coalitions:

- $v(\{a\}) = 0$; $v(\{b\}) = 0$; $v(\{a, b\}) = 2$; $v(\{a, c\}) = 2$;
- $v(\{b, c, d\}) = 0$; $v(\{a, b, c, d\}) = 7$;

Graph games

Several complexity results were provided in this setting:

- checking whether the core is non-empty is co-**NP**-complete
- checking whether an imputation is in the bargaining set is **NP**-hard
- a polynomial-time computable characterization for the Shapely value was provided
- the nucleolus was shown to coincide with the Shapely value, for non-negative components

Graph games

- But several questions were explicitly left open regarding solution concepts in the settings of both graph games and general compact coalitional games
- Although Deng and Papadimitriou's work has gained a prominent role through years, several of those questions have been left unanswered

Contribution I

We solved several of those open problems, by showing that, for graph games:

- Checking whether an outcome is in the kernel is Δ_2^P-complete;
- Checking whether an outcome is the nucleolus is Δ_2^P-complete; and,
- Checking whether an outcome is in the bargaining set is Π_2^P-complete.

Moreover, we have analyzed some generalizations and specializations of graph games

Contribution II

- Generalizations: for Bilbao's polynomial characteristic form games ($v(S)$ computed by an oracle requiring time polynomial in $|N|$)
  - we show that nothing has to be paid for this generality: checking membership in the kernel, the bargaining set or the nucleolus are still in $\mathbf{\Delta_2^P}$, $\mathbf{\Pi_2^P}$, and $\mathbf{\Delta_2^P}$, resp.
- Specializations: in graph g. having bounded tree-width, membership in the kernel is feasible in polynomial time

Some preliminary definitions

- A vector $(x_i)_{i \in N}$ (with $x_i \in \mathbb{R}$) is an *imputation* of $\mathcal{G}$ if $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$, for all $i \in N$
- The set of all the imputations of $\mathcal{G}$ is denoted by $X(\mathcal{G})$
- $\mathcal{I}_{i,j}$ is the set of all coalitions containing player i but not player j

The Kernel

- The *excess* $e(S, x)$ of S at the imputation $x \in X(\mathcal{G})$, is $v(S) - x(S)$, with $x(S) = \sum_{i \in S} x_i$
- The *surplus* $s_{i,j}(x)$ of i against j at x is $s_{i,j}(x) = \max_{S \in \mathcal{I}_{i,j}} e(S, x)$

Definition

The *kernel* $\mathscr{K}(\mathcal{G})$ of a game $\mathcal{G} = \langle N, v \rangle$ is the set: $\mathscr{K}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid s_{i,j}(x) > s_{j,i}(x) \Rightarrow x_j = v(\{j\}), \forall i, j \in N, i \neq j\}$.

The Bargaining set

- For an imputation x, (y, S) is an *objection of i against j to x* if $S \in \mathcal{I}_{i,j}$, $y(S) = v(S)$, and $y_k > x_k$ for all $k \in S$
- A *counterobjection to the objection* (y, S) *of i against j* is a pair (z, T) where $T \in \mathcal{I}_{j,i}$, $z(T) = v(T)$, and $z_k \geq x_k$ for all $k \in T \setminus S$ and $z_k \geq y_k$ for all $k \in T \cap S$
- If there is no counterobjection to (y, S), (y, S) is a *justified objection*.

Definition

The *bargaining set* $\mathscr{B}(\mathcal{G})$ of $\mathcal{G}$ is the set of all imputations to which there is no justified objection.

The Nucleolus

- For an imputation x, define the vector
$$\theta(x) = (e(S_1, x), e(S_2, x), \ldots, e(S_{2^n-1}, x))$$
with coalition excesses arranged in non-increasing order

- For imputations x, y, $\theta(x)$ *precedes* $\theta(y)$ ($\theta(x) \prec \theta(y)$), if $(\exists q)(\forall i < q)(\theta(x)[i] = \theta(y)[i] \land \theta(x)[q] < \theta(y)[q])$

Definition

The *nucleolus* $\mathcal{N}(\mathcal{G})$ of a game $\mathcal{G}$ is the set

$$\mathcal{N}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid \nexists y \in X(\mathcal{G}) \text{ s.t. } \theta(y) \prec \theta(x)\}^a$$

[a] For any game $\mathcal{G}$, $\mathcal{N}(\mathcal{G})$ is a singleton

Hardness results I: The kernel

Theorem

Let $\mathcal{G}$ be a graph game, and x an imputation of $\mathcal{G}$. Then deciding whether x belongs to $\mathscr{K}(\mathcal{G})$ is $\mathbf{\Delta_2^P}$-hard

Hardness of the Kernel: Proof sketch I

- Let $\phi = c_1 \wedge \ldots \wedge c_m$ be a **3CNF** satisfiable Boolean formula over the set of ordered variables $\{\alpha_1, \ldots, \alpha_n\}$
- The $\mathbf{\Delta_2^P}$-hard problem we use is establishing if $\alpha_1 = 1$ in the lexicographically-maximal assignment making ϕ true
- Based on ϕ, we build a graph $\boldsymbol{K}(\phi) = \langle (N_\mathbf{K}, E_\mathbf{K}), w \rangle$

Hardness of the Kernel: Proof sketch II

The nodes N_{K} (players):

- a *variable player* α_i, for each variable α_i in ϕ
- a *clause player* c_j, for each clause c_j in ϕ
- a *literal player* $\ell_{i,j}$ (either $\ell_{i,j} = \alpha_{i,j}$ or $\ell_{i,j} = \neg\alpha_{i,j}$), for each literal ℓ_i ($\ell_i = \alpha_i$ or $\ell_i = \neg\alpha_i$, respectively) as occurring in c_j
- two special players "*chall*" and "*sat*".

Hardness of the Kernel: Proof sketch III

The edges E_κ:

- **Positive edges**:
  - $\{c_j, \ell_{i,j}\}$, with $w(\{c_j, \ell_{i,j}\}) = 2^{n+3}$, for each literal ℓ_i occurring in c_j
  - $\{chall, \alpha_i\}$, with $w(\{chall, \alpha_i\}) = 2^i$, for each $1 \leq i \leq n$
  - $\{sat, \alpha_i\}$, with $w(\{sat, \alpha_i\}) = 2^i$, for each $2 \leq i \leq n$
  - $\{sat, \alpha_1\}$, with $w(\{sat, \alpha_1\}) = 2^1 + 2^0$

Hardness of the Kernel: Proof sketch IV

- **"Penalty" edges**:
  - $\{\ell_{i,j}, \ell_{i',j}\}$ with $w(\{\ell_{i,j}, \ell_{i',j}\}) = -2^{m+n+7}$, for each pair of literals ℓ_i and $\ell_{i'}$ occurring in c_j
  - $\{\alpha_{i,j}, \neg\alpha_{i,j'}\}$ with $w(\{\alpha_{i,j}, \neg\alpha_{i,j'}\}) = -2^{m+n+7}$, for each variable α_i occurring positively in c_j and negated in $c_{j'}$
  - $\{\alpha_i, \neg\alpha_{i,j}\}$ with $w(\{\alpha_i, \neg\alpha_{i,j}\}) = -2^{m+n+7}$, for each variable α_i occurring negated in c_j
- **"Normalizer" edge**: $\{chall, sat\}$ with $w(\{chall, sat\}) = 1 - \sum_{e \in E_{\mathbf{K}} | e \neq \{chall, sat\}} w(e)$

Hardness of the Kernel: Proof sketch V

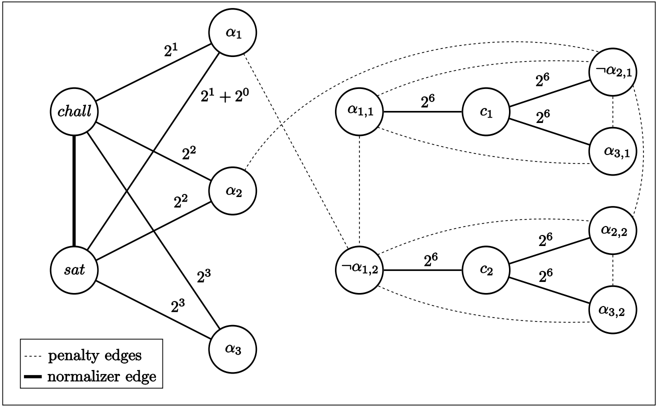


Figure: The game $\boldsymbol{K}(\widehat{\phi})$, where $\widehat{\phi} = (\alpha_1 \vee \neg\alpha_2 \vee \alpha_3) \wedge (\neg\alpha_1 \vee \alpha_2 \vee \alpha_3)$

Hardness of the Kernel: Proof sketch VI

- Consider x: $x_{sat} = 1$ and for all $i \neq sat$, $x_i = 0$; x is an imputation by the definition of $w(\{chal, sat\})$
- By definition of kernel, since *sat* is the only player for which $x_{sat} > v(\{sat\})$, $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ iff, for each player $i \neq sat$
$$\max_{S \in \mathcal{I}_{i,sat}} e(S, x) \leq \max_{S \in \mathcal{I}_{sat,i}} e(S, x)$$
- But, for each player $i \notin \{sat, chall\}$,
$$\max_{S \in \mathcal{I}_{i,sat}} e(S, x) \leq \max_{S \in \mathcal{I}_{sat,i}} e(S, x)$$
because $\{sat, chall\} \in \mathcal{I}_{sat,i}$

Hardness of the Kernel: Proof sketch VII

- Therefore, $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ iff
$$\max_{S \in \mathcal{I}_{chall,sat}} e(S, x) \leq \max_{S \in \mathcal{I}_{sat,chall}} e(S, x)$$
- By some calculations one finds that:
  - $\max_{S \in \mathcal{I}_{chall,sat}} e(S, x) = m \times 2^{n+3} + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i) = \texttt{true}} 2^i$
  - $\max_{S \in \mathcal{I}_{sat,chall}} e(S, x) = m \times 2^{n+3} + \max_{\sigma \models \phi} \left(|\{\alpha_1 \mid \sigma(\alpha_1) = \texttt{true}\}| + \sum_{\alpha_i | \sigma(\alpha_i) = \texttt{true}} 2^i \right) - 1$

Hardness of the Kernel: Proof skecth VIII

- Therefore, by substituting: $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ iff
$$1 + \max_{\sigma \models \phi} \sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i \leq$$
$$\max_{\sigma \models \phi} \left(\sum_{\alpha_i | \sigma(\alpha_i) = \mathtt{true}} 2^i + |\{\alpha_1 \mid \sigma(\alpha_1) = \mathtt{true}\}| \right)$$

- The last inequality being equivalent to $x \in \mathscr{K}(\boldsymbol{K}(\phi))$ if and only if α_1 is true in the lexicographically maximum satisfying assignment for ϕ

Hardness results II

Theorem

Let $\mathcal{G}$ be a graph game, and x an imputation of $\mathcal{G}$. Then:

- *deciding whether x belongs to $\mathscr{N}(\mathcal{G})$ is $\mathbf{\Delta_2^P}$-hard*
- *deciding whether x belongs to $\mathscr{B}(\mathcal{G})$ is $\mathbf{\Pi_2^P}$-hard*

Hardness results II: proof ideas

- Nucleolus: the proof also uses a reduction of the problem of deciding whether α_1 is true in the lexicographically maximum satisfying assignment of a given a **3CNF** Boolean formula
- Bargaining set: the proof uses a reduction of the problem of checking the validity of a quantified Boolean formula $\Phi = (\forall \boldsymbol{\alpha})(\exists \boldsymbol{\beta})\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$

Intermezzo

- Hardness results illustrated above are are tight (corresponding membership results can be established)
- We can do better, by proving membership results within the more general setting of *compact games*
- A class of games $\mathcal{C}$ is compact if, for every game $\mathcal{G} \in \mathcal{C}$, the game encoding (whose size is $||\mathcal{G}||$) includes the set N of all players (so that, $|N| \leq ||\mathcal{G}||$), and the function v is given by an oracle that computes $v(S)$ in time polynomial in $||\mathcal{G}||$.

Kernel: The membership theorem

Theorem

For any $\mathcal{G} \in \mathcal{C}_{cg}$, with $\mathcal{C}_{cg}$ compact: deciding whether an imputation x belongs to $\mathscr{K}(\mathcal{G})$ is feasible in $\mathbf{\Delta_2^P}$

Membership of the Kernel: Proof Idea

- we can compute in polynomial time the value $v(\{i\})$ for each player $i \in \{1, \ldots, n\}$
- for each pair of players i and j, compute $s_{i,j}(x)$:
  - representing $v(S)$ requires polynomially many bits
  - a binary search over the range of the values of the worth functions requires a polynomial number of steps
  - in **NP** we can check, for any value h in this range, whether there is a coalition S such that $v(S) - x(S) > h$, yielding $s_{i,j}(x)$
- therefore, it requires polynomially-many oracle calls to check that, for each pair of players i and j such that $x_j \neq v(\{j\})$, it holds that $s_{i,j}(x) \leq s_{j,i}(x)$.

Bargaining set: Some notes

- It was argued that telling an imputation to be in the bargaining s. is in $\mathbf{\Pi_2^P}$ for graph g. – guess an objection (**NP**) and check if a counterobjection exists (co-**NP**)
- This result holds, but it is restricted to games where values are represented with polynomially many bits
- We show that the membership in $\mathbf{\Pi_2^P}$ holds independently of the precision used to represent the reals in the game
- A characterization of a player *i* having a justified objection against a player *j* to *x* through *S* is preliminary

Bargaining set: A useful lemma

Lemma

Player i has a justified objection against player j to x through coalition $S \in \mathcal{I}_{i,j}$ iff there exists a vector $y \in \mathbb{R}^{|S|}$ such that:

a) $y(S) = v(S)$

b) $y_k > x_k$, *for each* $k \in S$

c) $v(T) < y(T \cap S) + x(T \setminus S), \forall T \in \mathcal{I}_{j,i}$.

Bargaining set: The membership theorem

Theorem

For any $\mathcal{G} \in \mathcal{C}_{cg}$, deciding whether an imputation x belongs to $\mathscr{B}(\mathcal{G})$ is feasible in $\mathbf{\Pi_2^P}$

Bargaining set: Proof sketch I

- The proof goes by showing that the complementary problem of deciding if $x \notin \mathscr{B}(\mathcal{G})$ is in $\mathbf{\Sigma_2^P}$
- One may guess two players i and j, and a coalition $S \in \mathcal{I}_{i,j}$, and then check if the system of inequalities $\mathtt{LP}$ of the previous lemma has a solution

Bargaining set: Proof sketch II

- For this last check, a co-**NP** oracle can be used:
  - $\texttt{LP}$ has $|S|$ variables $(y_1, \ldots, y_{|S|})$
  - by Helly's Theorem, for a collection $\mathcal{C} = \{c_1, \ldots, c_h\}$ of convex subsets of $\mathbb{R}^n$, $\bigcap_{c_i \in \mathcal{C}} c_i = \varnothing$ implies for a collection $\mathcal{C}' \subseteq \mathcal{C}$ to exist s.t. $|\mathcal{C}'| \leq n + 1$ and $\bigcap_{c_i \in \mathcal{C}'} c_i = \varnothing$
  - hence, if $\texttt{LP}$ has no solutions, there is a subset $\texttt{LP}'$ of $\texttt{LP}$ including $|S| + 1$ inequalities at most that has no solutions
  - therefore, one may guess $\texttt{LP}'$, and check in polynomial time that $\texttt{LP}'$ is infeasible

Nucleolus: The membership theorem

Lemma

For any $\mathcal{G} \in \mathcal{C}_{cg}$, computing $\mathscr{N}(\mathcal{G})$ is feasible in $\mathrm{F}\boldsymbol{\Delta}_2^P$.

Theorem

For any $\mathcal{G} \in \mathcal{C}_{cg}$, deciding if an imputation is in $\mathscr{N}(\mathcal{G})$ is in $\boldsymbol{\Delta}_2^P$.

Nucleolus: Proof idea

- We can show that it is possible to build in $\mathrm{F}\boldsymbol{\Delta}_2^P$ a sequence of short encodings of n linear programs, each of which depends on the previous element in the sequence
- And that the nucleolus of the given compact game is computable in polynomial time from the last element of this sequence

Some notes

- To date, we have a single tractability result to illustrate, regarding the kernel of a bounded treewidth graph game
- The result is proved by showing that computing the coalition over which the maximum excess at x is achieved can be expressed as an optimization problem over monadic second order logic for graph g. of bounded treewidth

The theorem

Theorem

Let $\mathcal{G} = \langle (N, E), w \rangle$ be a graph game such that (N, E) has tree-width bounded by k, and let x be an imputation of $\mathcal{G}$. Then, deciding whether $x \in \mathscr{K}(\mathcal{G})$ can be done in polynomial time.

Conclusion

- An account of the computational complexity of main solution concepts in compact coalitional games:
  - Several open problems regarding the setting of graph games have been answered
  - Several additional complexity results about generalizations of graph games have been provided
  - A tractability result regarding the kernel in graph games has been proved

Open problems

- Characterizing the tractability of solution concepts is interesting, within and outside the setting of graph games
- Other solution concepts pose other problems. A notable question is deciding whether a game has a Von-Neumann and Morgenstern solution (aka, stable set) or not

Conclusion

Many thanks for your kind attention