Introduction to coalitional games, infeasibility certificates and the Complexity of the Core

Luigi Palopoli

DEIS, Università della Calabria, Rende, Italy
palopoli@deis.unical.it

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Talk Outline

People that (as far as I know) are interested in game theory at Unical:

- F. Scarcello and G. Greco (Dip. Matematica) work on strategic games
- F. Scarcello, E. Malizia and L. Palopoli work on coalitional games (this talk....)
- G. Greco also works on auctions
- Active collaboration with G. Gottlob, D. Akatov (Oxford), T. Macini (U. Rome)

What is game theory about?

- Game theory consists of a set of formal tools helping in understanding how decision-makers interact.
- The basic assumption here is that decision-makers act rationally and take into account their knowledge about other decision makers
- Game theory comes in several guises (models) and has been exploited for the sake of analysis of rather diverse contexts

Game models

- Examples of Game models (and related applications) include:
  - Strategic games (analysis of political competitions)
  - Extensive games (analysis of bargaining and trades)
  - Repeated games (analysis of animal behavior formation through evolution)
  - Coalitional games (analysis of stable coalition formation in natural and synthetic societies)
- Each game model comes along with one or more solution concepts which have been defined in order to individuate the (form of the) outcomes of instances of that game model: for instance Nash equilibria are the reference solution concepts in strategic games

Classification of games

Groups of game models are distinguished by the nature of the interactions amongst involved players (the basic entity in all frameworks), for instance:

- Cooperative vs. Noncooperative: what are the primitive decisions? Those of individual players (Noncooperative) or those of groups of players taken as a whole (Cooperative)
- Strategic vs Extensive: what span has the decision horizon? It is just on point, so that all players' decisions are assumed to be taken simultaneously (Strategic) or it consists of a sequence of events, so that players can consider their plans of actions developed over time and contrasted with other players' previous decisions

Classification of games - continued.....

- Repeated vs. One-shot: how many times does this game will take place? Several times (So that players are possibly in the conditions to learn from past events or try to influence the other players' future behavior) (Repeated) or Only once (One-shot)
- with Perfect vs. Imperfect Information: Is each player perfectly informed about other players' moves? Yes Perfect information or No (Imperfect information)

Solution concepts

- Each game model comes along with one or more solution concepts which have been defined in order to individuate the (form of) the outcomes of instances of that game model: for instance Nash equilibria are the reference solution concepts in strategic games
- Solution concepts serves the purpose of capturing an arrangement of things that is someway stable, that is, it is immune by deviation as long as decision-makers act rationally
- In general, the number of possible outcomes that a solution concept associates with a given game may vary from 0 to infinite

Examples

The following table encodes the well known Prisoner's Dilemma strategic game:

	Confess	Don't Confess
Confess	-3,-3	0,-4
Don't Confess	-4,0	-1,-1

The entries represent the payoffs of the first and second player, resp. (say, the sentenced years in prison). The optimal solution would clearly be the pair of actions *(Don't Confess; Don't Confess)*, but it is not stable, since both players have an incentive to change their state. The only Nash equilibrium here is *(Confess; Confess)*

The following table encodes another well known strategic games, known as Battle of Sexes (aka, *Back or Stravinsky)*.

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

In this case, there are two Nash equilibria, namely, *(Bach; Bach)* and *(Stravinsky; Stravinsky)*.

What is a coalitional game

- Coalitional games model situations where groups of players (Coalitions) can cooperate in order to obtain a certain worth
- Worths are assigned to coalitions and the outcome of a coalitional game is the specification of the coalition that forms and the (joint) actions it takes
- Coalition formation is determined by individual players' preference profiles over the set of possible outcomes of a game

What is a coalitional game

Therefore, a coalitional games is defined, in general, by specifying

- the set N of players in the game
- a function v, defined on the 2^N, returning the worth assigned to any coalition $s \subseteq N$
- preference relations of players in N over possible outcomes

The coalition N including all players is called the grand coalition

Applications

- Coalitional games have been extensively used to study applicative scenarios in economics and social sciences (market structure analysis, voting systems,....)
- In computer science, coalitional games are relevant, for instance, to:
  - distributed AI
  - multi-agent systems
  - electronic commerce
  - modeling and protocol design in large networks

Example

Multiple users want to route network traffic through a switch, which has a flow-dependent delay (cost). The queueing delay cost has to be shared among the users. This can be modeled as a coalitional game, where a suitable solution concept can be chosen and exploited to correspond to fair cost sharing.

Kinds of coalitional games

Coalitional games come into two main guises, depending on whether the worth of a coalition can be freely distributed amongst its members or not:

- Games with transferable payoffs (TU-Games), where the worths are transferable amongst players forming a coalition without any limitation
- Games with non-trasfereable payoffs (NTU-Games), otherwise

TU-games

Definition

A Coalitional Game with transferable payoffs is a pair $\langle N, v\rangle$ where

- *N is the finite set of players;*
- *v is a function that associates with every coalition s a real number $v(s)$ (the worth of s) ($v : 2^N \rightarrow \Re$).*

TU-games

- In studying coalitional TU-games, it is assumed that the grand coalition forms, for otherwise it would be meaningless to analyze fairness or stability conditions on distributions of payoffs among its members
- This assumption can be imposed by requiring the game to be cohesive

Definition

A TU-game $\langle N, v\rangle$ is cohesive if

$$v(N) \geq \sum_{k=1}^{P} v(s_k) \text{ for each partition } \{s_1, \ldots, s_P\} \text{ of } N$$

Payoff profiles

- Distributions of payoffs amongst the member of a coalition are described by vectors of reals
- For any coalition, consistent distributions are those where the sum of the distributed payoffs equals the worth assigned to that coalition

Definition

Let $n = |N|$. A profile $\bar{x}$ for N is a vector of reals $(\bar{x}_1, \ldots, \bar{x}_n)$. For a coalition $s \subseteq N$, define $\bar{x}(s) \equiv \sum_{i \in s} \bar{x}_i$. Then, $\bar{x}$ is said a s-feasible payoff profile if $\bar{x}(s) = v(s)$. Moreover, $\bar{x}$ is said a feasible payoff profile if it is an N-feasible payoff profile.

Example

An expedition of n people discover a treasure. It requires two people to carry out one piece of the treasure, in which case the value of the carried piece is equally shared between the two. For a subset s of people, the worth is given by

$$v(s) = \lfloor \frac{|s|}{2} \rfloor$$

If $|N| = 2$, $(1/2, 1/2)$ is a stable sharing. What if $|N| = 3$? Clearly $(1/2, 1/2, 0)$ is not stable, nor is $(1/3, 1/3, 1/3)$. In this case, there is no stable sharing. In general, there is no stable sharing for |N| odd, whereas the profile $(1/2, 1/2, \ldots)$ is stable for $|N|$ even.

NTU-games

Definition

A Coalitional Game without transferable payoff is a four-tuple $\langle N, X, v, (\succsim_i)_{i \in N} \rangle$, where:

- *N is a finite set of players;*
- *X is the set of all possible consequences;*
- *$v : s \to 2^X$ is a function that assigns, to any coalition $s \subseteq N$ of players, a set of consequences $v(s) \subseteq X$;*
- *$(\succsim_i)_{i \in N}$ is the set of all preference relations $\succsim_i$ on X, $\forall i \in N$.*

Solution concepts

- As in the general case, a solution concept assigns to each coalitional game a set of possible outcomes, hereby capturing a rational behavior of decision makers (the players) participating into the given game
- The stability condition, in this context, requires that the produced arrangement be immune by deviations caused by groups of players (by contrast in strategic game solutions, for instance, deviations are determined by individual players)

Solution concepts

- A relevant number of solution concepts have been defined for both TU and NTU coalitional games
- Most often, solution concept definitions are given along a set of axioms, that are proved "equivalent" with the defined solution concept

Solution concepts

- A (rather partial) list of solution concepts follows:
  - the stable set
  - the core
  - the Shapley value
  - the Banzhaf index
  - the bargaining set
  - the Kernel
  - the nucleoulus
- In the following, we are going to introduce some of these concepts and then focus our analysis on the core

Stable sets

- Stable sets, defined by Von Neumann and Morgenstern in 1944, are one of the oldest and best established of the solution concepts
- Each stable set Y includes distributions of the worth such that none of the members of Y is preferable to the other and each distribution not included in Y has a preferable distribution that is in Y

Stable sets

An *imputation* x is a feasible payoff profile such that $x_i >= v(\{i\})$. Let X_G be the set of imputations of the game G.

Definition

An imputation x dominates an imputation y via s, written $x \succ_s y$ if $(\forall i \in s)(x_i > y_i)$ and $x(s) \leq v(s)$. Let

$$D(Y) = \{z \in X_G \mid (\exists s)(\exists y \in Y)(y \succ_s z)\}$$

Stable sets

Definition

A stable set $Y \subseteq X_G$ is a set of imputations such that

$$Y = X_G \setminus D(Y)$$

Some properties of stable sets

- A game may have 0, 1 or more stable sets
- No stable set is a proper subset of another stable set

Bargaining sets

Let x be an imputation of a TU-game $< N, v >$. Define objections and counterobjections as follows.

- (y, S), where y is $S-$ *feasible* is an *objection* of i against j if $i \in S$, $j \notin S$, $y_k > x_k$ for all $k \in S$
- (z, T), where z is $T-$ *feasible* is a *counter-objection* to the objection (y, S) of i against j if $j \in T$, $i \notin T$, $z_k \geq x_k$ for all $k \in S - T$ and $z_k \geq y_k$ for all $k \in S \cap T$

Bargaining sets

Definition

The Bargaining set *of a TU-game* $< N, v >$ *is the set of all imputations* x *such that for any objection* (y, S) *of any player* i *against any player* j *there is a corresponding counter-objection of* j *against* (y, S).

The Kernel

Let x be an imputation of a TU-game $< N, v >$. For any coalition S, define $e(S, x) = v(S) - x(S)$.
Define objections and counterobjections as follows.

- S is an *objection* of i against j to x if $i \in S$, $j \notin S$ and $x_j > v(\{j\})$.
- $T$is a *counter-objection* to the objection S of i against j if $j \in T$, $i \notin T$, $e(T, x) \geq e(S, x)$.

The Kernel

Definition

The Kernel *of a TU-game* $< N, v >$ *is the set of all imputations* x *such that for any objection* S *of any player* i *against any player* j *there is a corresponding counter-objection of* j *against* S*.*

The Kernel is always contained in the bargaining set and is non-empty.

The Shapley value

For a TU-game $< N, v >$, for any player i, for a coalition S s.t. $i \notin S$, define $\Delta_i = v(S \cup \{i\}) - v(S)$.

Definition

The Shapley value ϕ *is defined as follows:*
$\phi_i(N, v) = (1/|N|!) \sum_{r \in R} \Delta_i(S_i(r))$
for all i, where R is the set of all orderings of N and $S_i(r)$ is the set of players preceding i in the ordering r.

The core

- The core for coalitional games can be seen as an analogous of the Nash equilibrium for strategic games and it is probably the most important solution concept defined for such games
- The core forces distributions that are “stable”, i.e., no subsets of players improve their worths by leaving the grand-coalition
- Two definitions are provided next, one for TU-games, the latter for NTU-games

The core in TU-games

Definition

The core of a coalitional game with transferable payoffs $\langle N, v \rangle$ is the set of all feasible payoff profiles $\bar{x}$ such that, for all coalitions $s \subseteq N$, $\bar{x}(s) \geq v(s)$.

It follows that the core is the n-dimensional hyperspace defined by the following 2^n inequalities:

$$\sum_{i \in s} x_i \geq v(s), \quad \forall s \subseteq N \wedge s \neq \emptyset$$
$$\sum_{i \in N} x_i \leq v(N),$$

where the last inequality enforces the feasibility of profiles.

The core in NTU-games

Definition

The *core of the coalitional game without transferable payoffs* $\langle N, X, v, (\succsim_i)_{i \in N} \rangle$ is the set of all $\bar{x} \in v(N)$ such that there is no coalition $s \subseteq N$ with a $\bar{y} \in v(s)$ such that $\bar{y} \succ_i \bar{x}$ for all $i \in s$.

Some properties of the core (and stable sets)

- The core may or may not exist for a given game
- Conditions have been defined for the existence of the core (cf., Bondareva-Shapley theorem)
- The core is the set of undominated imputations: $\{x \in X_G \mid (\nexists s)(\nexists y \in X_G) y \succ_s x\}$
- It then follows that:
  - The core is a subset of every stable set
  - If the core is a stable set, then it is the only stable set

How to represent a coalitional game?

- Explicitly representing v not feasible for large games (e.g., with Internet application modeling)
- Compact representation of the worth function v needed with input size being (more or less) as large an $|N|$

How to represent a coalitional game?

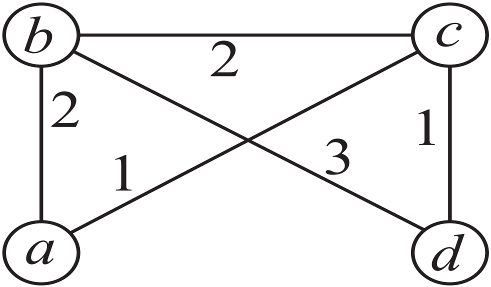
Several proposals, including:

- Marginal Contribution Nets:
  - Games represented using set of rules *pattern* $\rightarrow$ *value*;
  - Example: for $N = \{a, b\}$, $v(\{a\}) = 0$, $v(\{b\}) = 2$, $v(\{a, b\}) = 7$:

$$\{b\} \rightarrow 2 \qquad \{a \wedge b\} \rightarrow 5$$

- Games on Graphs:
  - Players are graph vertices
  - $v(s)$ is the sum of arc weights in the subgraph induced by s.

An example of graph game



Worths for some sample coalitions:

- $v(\{a\}) = 0$; $v(\{b\}) = 0$; $v(\{a,b\}) = 2$; $v(\{a,c\}) = 1$;
- $v(\{b,c,d\}) = 6$; $v(\{a,b,c,d\}) = 9$;

A general Framework for Compact Representations

Our most general result refers to a new compact representation scheme just requiring the worth function to be computable in FNP. We begin with a simpler setting though.

The FP Compact Representations

- $\mathcal{C}$ class of TU-games as defined by a given encoding scheme
- The *consequence* relation for $\mathcal{C}$ is the set of tuples $W_{\mathcal{C}} = \{\langle \mathcal{G}, s, w \rangle \mid \mathcal{G} \in \mathcal{C}, v_{\mathcal{G}}(s) = w\}$

The FP Compact Representations

$\mathcal{C}(\mathcal{R})$ class of TU-games as defined by the encoding $\mathcal{R}$

Definition

$W_{\mathcal{C}}$ is polynomial-time computable $\exists M$ (a PTIME deterministic transducer) that, given any $\mathcal{G} \in \mathcal{C}(\mathcal{R})$ and $s \subseteq N$, returns w such that $\langle \mathcal{G}, s, w \rangle \in W_{\mathcal{C}(\mathcal{R})}$ in at most $||\langle \mathcal{G}, s \rangle||^k$ steps, with k a positive integer.

$\mathcal{R}$ is called FP-representation if $W_{\mathcal{C}}$ is polynomial-time computable.

Complexity of the core

- For the mentioned compact representations, checking whether the core is not empty is $\mathrm{co\text{-}NP}$-hard
- However, membership in $\mathrm{co\text{-}NP}$ is not easily established and it was left as an open problem by several authors

Complexity of the core

- We provide a rather complete answer for TU-games (and for NTU-games as well)
- The membership is settled for all FP-representations (generalized to FNP-representations as well)
- In the following results, we assume an FP-representation of games even though not explicitly stated

The main result

The $\mathrm{co\text{-}NP}$-hardness of core non-emptyness in TU-games follows from known results. Our first main result is summarized in the following theorem:

Theorem

Let $\mathcal{R}$ be a FP-representation. Given any TU-game $\mathcal{G} \in \mathcal{C}(\mathcal{R})$, deciding whether the core of $\mathcal{G}$ is not empty is in $\mathrm{co\text{-}NP}$.

A corollary

Hereby the precise complexity of the core non-emptiness problem for marginal contribution nets (left open by Ieong and Shoham in their ACM EC'05 paper) is settled as well:

Corollary

For TU-games encoded as marginal contribution nets, deciding whether the core is not empty is $\mathrm{co\text{-}NP}$*-complete.*

Proof sketch

The core is the n-dimensional hyperspace defined by the following 2^n inequalities:

$$\sum_{i \in s} x_i \geq v(s), \quad \forall s \subseteq N \wedge s \neq \emptyset \tag{3.1}$$

$$\sum_{i \in N} x_i \leq v(N), \tag{3.2}$$

(the last inequality enforcing the feasibility of profiles)

Proof sketch

- The core of a TU game with n players is a polyhedral set of $\Re^n$.
- Proof intuition: any TU-game with an empty core has a small infeasibility certificate for it.

Proof sketch

- Coalitions correspond to the inequalities (3.1) and hence with the associated half-spaces of $\Re^n$
- The intersection of the half-spaces associated with a set of coalitions (inequalities) S is denoted $\mathrm{Pol}(S)$

Definition

Let $\mathcal{G} = \langle N, v \rangle$ be a TU-game. A set of coalitions $S \subseteq 2^N$ is a *certificate of emptiness* (or *infeasibility certificate*) for the core of $\mathcal{G}$ if the intersection of $\mathrm{Pol}(S)$ with the grand-coalition halfspace (3.2) is empty.

Proof sketch

- Let P be the polyhedron of $\Re^n$ obtained as the intersection of all halfspaces (3.1)
- Since S is a subset of the family of all possible coalitions, $P \subseteq \mathrm{Pol}(S)$
- If $\mathrm{Pol}(S)$ has empty intersection with the grand-coalition halfspace (3.2), P has an empty intersection with it as well

Proof sketch

Our theorem follows from:

Theorem

Let $\mathcal{G} = \langle N, v \rangle$ be a TU-game. If the core of $\mathcal{G}$ is empty, there is a certificate of emptiness S for it such that $|S| \leq |N|$.

Proof sketch - FP-representation of TU games

For a TU-game $\mathcal{G} = \langle N, v \rangle$ with a FP-representation, a NdTM checks that the core is empty in PTIME by:

- guessing the set S
- computing (in PTIME) the worth $v(s)$, for each $s \in S$ and for the grand-coalition N
- checking that $\mathrm{Pol}(S) \cap H_P^+ = \emptyset$, where H_P^+ is the grand-coalition halfspace (3.2), which is tantamount to solving a linear system consisting of $n + 1$ inequalities.

...recalling....NTU-games

In NTU-games payoffs cannot be freely distributed:

Definition

A Coalitional Game without transferable payoff is a four-tuple $\langle N, X, v, (\succsim_i)_{i \in N} \rangle$, where:

- *N is a finite set of players;*
- *X is the set of all possible consequences (allowed distributions);*
- *$v : s \rightarrow 2^X$ assigns to $s \subseteq N$, a set of consequences $v(s) \subseteq X$;*
- *$(\succsim_i)_{i \in N}$ is the set of all preference relations $\succsim_i$ on X, $\forall i \in N$.*

The NTU-game case

- NTU-games represent a generalization of TU-games
- In NTU-games the allowed distributions of the worth are fixed a-priori with the game
- a TU-game is simply a NTU-game where the allowed distributions comprise all the possible distributions of worths

...and the core in NTU-games

Definition

The *core of the NTU-game* $\langle N, X, v, (\succsim_i)_{i \in N} \rangle$ is the set of all $\bar{x} \in v(N)$ such that there is no coalition $s \subseteq N$ with a $\bar{y} \in v(s)$ such that $\bar{y} \succ_i \bar{x}$ for all $i \in s$.

The complexity of the core for NTU-games

Checking core non-emptyness for NTU-games has the same complexity as for TU-games when FP-representations are considered (here we require the PTIME transducer to return all the consequences for the given coalition)

Theorem

Let $\mathcal{R}$ be a FP-representation and $\mathcal{G} \in \mathcal{C}(\mathcal{R})$ be a NTU-game. Deciding whether the core of $\mathcal{G}$ is not empty is $\mathrm{co\text{-}NP}$-complete

- The constraint that games are encoded using a $\mathcal{FP}$-representation scheme can be relaxed
- The representations defined next are captured via non-deterministic PTIME trasducers

A general Framework for Compact Representations

Definition

A worth (consequence) relation $W_{\mathcal{C}}$ is k-balanced if $||w|| \leq ||\langle \mathcal{G}, s \rangle||^k$. $W_{\mathcal{C}}$ is said k-decidable if there is a non-deterministic Turing machine that decides $W_{\mathcal{C}}$ in at most $||\langle \mathcal{G}, s, w \rangle||^k$ time.

Therefore, a non-deterministic Turing transducer M exists that computes in $O(||\langle \mathcal{G}, s \rangle||^k)$ time the worth $v(s)$ (resp. some consequences in $v(s)$) of any coalition s

$\mathcal{C}(\mathcal{R})$ class of games as defined by a compact encoding $\mathcal{R}$

Definition

The relation $W_{\mathcal{C}(\mathcal{R})}$ is non-deterministically polynomial-time computable if there is a positive integer k such that $W_{\mathcal{C}(\mathcal{R})}$ is k-balanced and k-decidable.

$\mathcal{R}$ is called FNP-representation if $W_{\mathcal{C}(\mathcal{R})}$ is non-deterministically polynomial-time computable

When FNP-representations are considered, the complexity of checking core non-emptyness remains unchanged for TU-games, whereas jumps one level up in PH for NTU-games

Theorem

Let $\mathcal{R}$ be a FNP-representation and let $\mathcal{G} \in \mathcal{C}(\mathcal{R})$. Deciding whether the core of $\mathcal{G}$ is not empty is:

- Σ_2^P*-complete, if $\mathcal{G}$ is a NTU-game*
- $\mathrm{co\text{-}NP}$*-complete, if $\mathcal{G}$ is a TU-game*

The Complexity of Core non-emptiness

A summary of the complexity results:

	FP representation (e.g. MC Nets, Games on Graphs)	FNP representation (e.g. NTU MC Nets)
TU Games	co-NP-complete	co-NP-complete
NTU Games	co-NP-complete	Σ_2^P-complete

Conclusion

- Games in coalitional form are relevant to many application domains
- The core is the most important of the solution concepts defined for coalitional games
- When many players are involved (e.g., with Internet applications), the worth function is to be compactly represented

Conclusion

- We have determined the complexity of core non-emptyness for games in compact representation form for both transferable and non-trasferable utilities
- The representation is simply constrained to be computable in FNP, which is quite a weak requirement allowing to capture most compact representation schemes used to date

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Conclusion

Thanks!